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Exactly solvable toy model for the pseudogap state

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Abstract. We present an exactly solvable toy model which describes the emergence of a pseudogap in an electronic system due to a fluctuating off-diagonal order parameter. In one dimension our model reduces to the fluctuating gap model (FGM) with a gap $\Delta(x)$ that is constrained to be of the form $\Delta(x) = A\mathrm{e}^{\mathrm{i}Qx}$, where A and Q are *random variables*. The FGM was introduced by Lee, Rice and Anderson [Phys. Rev. Lett. **31**, 462 (1973)] to study fluctuation effects in Peierls chains. We show that their perturbative results for the average density of states are exact for our toy model if we assume a Lorentzian probability distribution for Q and ignore amplitude fluctuations. More generally, choosing the probability distributions of A and Q such that the average of $\Delta(x)$ vanishes and its covariance is $\langle\Delta(x)\Delta^*(x')\rangle = \Delta_{\mathrm{s}}^2 \exp[-|x-x'|/\xi]$, we study the combined effect of phase and amplitude fluctuations on the low-energy properties of Peierls chains. We explicitly calculate the average density of states, the localization length, the average single-particle Green's function, and the real part of the average conductivity. In our model phase fluctuations generate delocalized states at the Fermi energy, which give rise to a finite Drude peak in the conductivity. We also find that the interplay between phase and amplitude fluctuations leads to a weak logarithmic singularity in the single-particle spectral function at the bare quasi-particle energies. In higher dimensions our model might be relevant to describe the pseudogap state in the underdoped cuprate superconductors.

PACS. 71.23.-k Electronic structure of disordered solids – 02.50.Ey Stochastic processes – 71.10.Pm Fermions in reduced dimensions (anyons, composite fermions, Luttinger liquid, etc.)

1 Introduction

The physical origin of the pseudogap behavior observed in the normal state of the high-temperature cuprates is still controversial. Several mechanisms have been proposed. According to Schmalian *et al.* [1] the normal state of the underdoped cuprates can be modeled by a nearly antiferromagnetic Fermi liquid, and the experimentally observed pseudogap behavior is closely related to strong antiferromagnetic spin fluctuations. An alternative explanation which has been advanced by Emery and Kivelson [2] relates the pseudogap behavior to precursor superconducting fluctuations. In this scenario thermal fluctuations of the phase of the superconducting order parameter are responsible for a destruction of superconductivity above the transition temperature T_{c}. However, in a wide range of temperatures $T > T_{\mathrm{c}}$ the local amplitude of the superconducting gap is finite. In this paper we shall propose a simple exactly solvable phenomenological model which describes the destruction of phase coherence due to phase and amplitude fluctuations of the superconducting order parameter in the pseudogap state.

To study superconducting fluctuations in a normal metal one can start with the Gorkov equation for the 2×2 matrix Green's function for electrons with energy dispersion $\epsilon(\mathbf{k})$ that are coupled to a space-dependent complex pairing field $\Delta(\mathbf{r})$ [3],

$$[\omega - \hat{H}_{\mathbf{r}}]\, \mathcal{G}^{(d=3)}(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}')\sigma_0 \;, \tag{1}$$

$$\hat{H}_{\mathbf{r}} = \begin{pmatrix} \epsilon(-\mathrm{i}\nabla_{\mathbf{r}}) - \mu & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & \epsilon(\mathrm{i}\nabla_{\mathbf{r}}) - \mu \end{pmatrix} \;. \tag{2}$$

Here, σ_0 is the 2×2 unit matrix and μ is the chemical potential. In the absence of true superconducting long-range order the pairing field $\Delta(\mathbf{r})$ can be considered as a random variable with zero average and correlations that fall off exponentially with distance,

$$\langle\Delta(\mathbf{r})\rangle = 0 \;, \tag{3}$$

$$\langle\Delta(\mathbf{r})\Delta^*(\mathbf{r}')\rangle \equiv \frac{\int \mathcal{D}\{\Delta\}\mathrm{e}^{-S\{\Delta\}}\Delta(\mathbf{r})\Delta^*(\mathbf{r}')}{\int \mathcal{D}\{\Delta\}\mathrm{e}^{-S\{\Delta\}}} = \Delta_{\mathrm{s}}^2 \mathrm{e}^{-|\mathbf{r}-\mathbf{r}'|/\xi} \;. \tag{4}$$

Here, $S\{\Delta\}$ is the Ginzburg-Landau functional of the order parameter field, ξ is the correlation length, and the energy scale Δ_{s} characterizes the strength of the correlations.

To simplify the algebra and to make contact with other theoretical work on pseudogap physics, we shall focus in this work on the semiclassical limit of the Gorkov

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equation, which are related to the so-called Andreev equation [4]. In the weak coupling limit, where $|\Delta(\mathbf{r})|$ is small compared with the chemical potential, we may linearize the energy dispersion in equation (1) for wavevectors $\mathbf{k}$ close to the Fermi surface, provided we are only interested in long-wavelength, low-energy properties of the system. In the semiclassical limit it is useful to decompose the position vector as $\mathbf{r} = x\mathbf{n}+\mathbf{r}_\perp$ where $\mathbf{n}$ is a unit vector in the direction of the momentum of the electron, and $\mathbf{r}_\perp$ is orthogonal to $\mathbf{n}$. Writing $\partial_x = \mathbf{n}\cdot\nabla_\mathbf{r}$, equations (1, 2) can be replaced by an effective one-dimensional problem [4]

$$[\omega - \hat{H}_x]\,\mathcal{G}(x,x',\omega) = \delta(x-x')\sigma_0\ , \tag{5}$$

$$\hat{H}_x = \begin{pmatrix} -\mathrm{i}v_\mathrm{F}\partial_x & \Delta(x) \\ \Delta^*(x) & \mathrm{i}v_\mathrm{F}\partial_x \end{pmatrix}\ . \tag{6}$$

We shall refer to equation (6) as the Hamiltonian of the fluctuating gap model (FGM). All quantities depend now parametrically on $\mathbf{r}_\perp$ and $\mathbf{n}$. Physical observables should be averaged over all directions of $\mathbf{n}$. In this paper we shall only consider the effective one-dimensional problem defined by equations (5, 6). We require that the first and the second moments of the fluctuating gap $\Delta(x)$ are given by

$$\langle\Delta(x)\rangle = 0\ , \tag{7}$$

$$\langle\Delta(x)\Delta^*(x')\rangle = \Delta_\mathrm{s}^2\mathrm{e}^{-|x-x'|/\xi}\ . \tag{8}$$

In the following, we shall construct a special non-Gaussian probability distribution of $\Delta(x)$ satisfying equations (7, 8) for which equation (5) can be solved exactly. Moreover, as will be briefly discussed in Section 4, it is straightforward to generalize our model to dimensions $d > 1$ and to arbitrary energy dispersions $\epsilon(\mathbf{k})$, although the calculation of physical quantities becomes more tedious.

Apart from its relevance in the semiclassical theory of superconductivity, the problem defined by equations (5–8) describes also the low-energy physics in quasi-one-dimensional Peierls and spin-Peierls systems [5,6]. Lee, Rice and Anderson [5] used this model to study fluctuation effects close to the Peierls transition. In this case $\Delta(x)$ can be identified with the fluctuating Peierls order parameter, and the two diagonal elements in our Hamiltonian (6) represent the kinetic energy of the electrons in the vicinity of the two Fermi points $\pm k_\mathrm{F}$. Physical quantities should again be averaged over the probability distribution of $\Delta(x)$, which can be obtained from the Ginzburg-Landau expansion [5]. Within the Gaussian approximation, the truncated Ginzburg-Landau functional in the disordered phase is of the form

$$S\{\Delta\} = \int\frac{\mathrm{d}q}{2\pi}\,\frac{1+q^2\xi^2}{2\Delta_\mathrm{s}^2\xi}\,\Delta_q^*\Delta_q\ , \tag{9}$$

where

$$\Delta_q = \int\mathrm{d}x\,\mathrm{e}^{-\mathrm{i}qx}\Delta(x)\ . \tag{10}$$

One easily verifies that equations (7, 8) are indeed satisfied. Note that for commensurate Peierls chains the order parameter field can be chosen real, while it is complex for incommensurate chains. In this work we shall focus on the incommensurate case, where zero-energy states and the associated Dyson singularities are absent [7,8]. Lee, Rice and Anderson treated the effect of the order parameter fluctuations on the average electronic density of states (DOS) $\langle\rho(\omega)\rangle$ within the Born approximation. Within this approximation one finds that, in the regime where the dimensionless parameter

$$\bar{\gamma} \equiv \frac{v_\mathrm{F}}{2\Delta_\mathrm{s}\xi} \tag{11}$$

is small compared with unity, the DOS develops a pseudogap for $|\omega| \lesssim \Delta_\mathrm{s}$, with a minimum given by [9]

$$\frac{\langle\rho(0)\rangle^\mathrm{pert}}{\rho_0} = \frac{\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}}\ . \tag{12}$$

Here,

$$\rho_0 = \frac{1}{\pi v_\mathrm{F}} \tag{13}$$

is the DOS for $\Delta(x) = 0$, which is a constant due to the linearization of the energy dispersion. Note that equation (12) predicts for $\bar{\gamma} \ll 1$ to leading order

$$\frac{\langle\rho(0)\rangle^\mathrm{pert}}{\rho_0} \sim \bar{\gamma} \propto \xi^{-1}\ , \tag{14}$$

which disagrees with a non-perturbative result by Sadovskii [10], who found for the model defined by equations (5–8) for a Gaussian distribution of $\Delta(x)$

$$\frac{\langle\rho(0)\rangle^\mathrm{Sadovskii}}{\rho_0} \approx 0.541[2\bar{\gamma}]^{1/2} \propto \xi^{-1/2}\ . \tag{15}$$

However, the algorithm constructed by Sadovskii [10] is not exact [7,11], so that it is not clear whether equation (15) is correct or not. To clarify this point, we have recently developed an exact numerical algorithm for calculating the DOS of the FGM [8]. For a Gaussian distribution of $\Delta(x)$ with zero average and covariance given by equation (8) the result is

$$\frac{\langle\rho(0)\rangle^\mathrm{Gauss}}{\rho_0} \approx a[2\bar{\gamma}]^b \propto \xi^{-b}\ , \tag{16}$$

where

$$a = 0.6397 \pm 0.0066\ \ ,\ \ b = 0.6397 \pm 0.0024\ . \tag{17}$$

Hence, for Gaussian disorder with a finite correlation length both perturbation theory and Sadovskii's algorithm do not give the correct ξ-dependence of the average DOS at the Fermi energy. Another attempt to investigate the discrepancy between equations (12, 15) numerically was recently made by Millis and Monien [12]. They found for the exponent b in equation (16) a value between 2/3 and 1, which is outside our error-bars in equation (17). Note,

$$\mathcal{G}(q,q',\omega) = \begin{pmatrix} \dfrac{2\pi\delta(q-q')[\omega-2\eta+v_{\mathrm{F}}q]}{[\omega-2\eta+v_{\mathrm{F}}q][\omega-v_{\mathrm{F}}q]-|A|^2} & \dfrac{2\pi\delta(q-q'-Q)A}{[\omega-2\eta+v_{\mathrm{F}}q][\omega-v_{\mathrm{F}}q]-|A|^2} \\ \dfrac{2\pi\delta(q-q'+Q)A^*}{[\omega-2\eta-v_{\mathrm{F}}q][\omega+v_{\mathrm{F}}q]-|A|^2} & \dfrac{2\pi\delta(q-q')[\omega-2\eta-v_{\mathrm{F}}q]}{[\omega-2\eta-v_{\mathrm{F}}q][\omega+v_{\mathrm{F}}q]-|A|^2} \end{pmatrix} . \tag{26}$$

however, that Millis and Monien studied a lattice regularization of the continuum model (6), and no attempt was made to carefully relate the bare parameters that appear in the lattice and the continuum models. In this work we shall show that the exponent characterizing the behavior of the DOS at the Fermi energy on ξ is non-universal in the sense that it depends on the precise form of the probability distribution of the fluctuating gap. In particular, the non-Gaussian terms in the Ginzburg-Landau functional can change the numerical value of this exponent, so that the behavior given in equations (16, 17) can only be expected to be correct for Gaussian disorder.

Finally, it should be mentioned that a generalization of the model defined in equations (5–8) has been used in reference [1] to explain the pseudogap behavior in the cuprates within antiferromagntic Fermi liquid theory. Then the scalar field $\Delta(x)$ should be replaced by a matrix field $\sum_i S_i(x)\sigma_i$, where σ_i are the Pauli matrices, and the fields $S_i(x)$ represent the components of the antiferromagnetic spin density field. In fact, the recent interest in the non-perturbative approach invented many years ago by Sadovskii [10] is motivated by its possible relevance to the cuprate superconductors.

2 Exact Green's function of the fluctuating gap model for $\Delta(x) = Ae^{iQx}$

In this section we shall solve equation (5) exactly for a special form of the probability distribution of $\Delta(x)$ which is constructed such that its covariance is given by equation (8). To begin with, let us perform the following gauge transformation [13],

$$\mathcal{G}(x,x',\omega) = \mathrm{e}^{\frac{\mathrm{i}}{2}\alpha(x)\sigma_3}\tilde{\mathcal{G}}(x,x',\omega)\mathrm{e}^{-\frac{\mathrm{i}}{2}\alpha(x')\sigma_3} , \tag{18}$$

where the gauge function $\alpha(x)$ will be specified shortly. From equation (5) we find that the transformed Green's function $\tilde{\mathcal{G}}(x,x',\omega)$ satisfies

$$\Big[\omega - \frac{v_{\mathrm{F}}}{2}\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x} + \mathrm{i}v_{\mathrm{F}}\partial_x\sigma_3 - \Delta(x)\mathrm{e}^{-\mathrm{i}\alpha(x)}\sigma_+ - \Delta^*(x)\mathrm{e}^{\mathrm{i}\alpha(x)}\sigma_-\Big]\tilde{\mathcal{G}}(x,x',\omega) = \delta(x-x')\sigma_0 . \tag{19}$$

Suppose now that $\Delta(x)$ is of the form

$$\Delta(x) = A\mathrm{e}^{\mathrm{i}Qx} , \tag{20}$$

where A and Q are both random but independent of x. Then the x-dependence of $\Delta(x)$ in equation (19) can be removed by choosing $\alpha(x) = Qx$. Moreover, with this choice the second term on the left-hand side of equation (19) reduces to a constant

$$\frac{v_{\mathrm{F}}}{2}\frac{\mathrm{d}\alpha(x)}{\mathrm{d}x} = \frac{v_{\mathrm{F}}Q}{2} \equiv \eta , \tag{21}$$

so that

$$[\omega - \eta + \mathrm{i}v_{\mathrm{F}}\partial_x\sigma_3 - A\sigma_+ - A^*\sigma_-]\,\tilde{\mathcal{G}}(x,x',\omega) = \delta(x-x')\sigma_0 . \tag{22}$$

Thus, a phase of the order-parameter varying linearly in space can be absorbed by a finite shift of the energy. Equation (22) is translational invariant and is easily solved by a Fourier transformation,

$$\tilde{\mathcal{G}}(x,x',\omega) = \int\frac{\mathrm{d}q}{2\pi}\mathrm{e}^{\mathrm{i}q(x-x')}\tilde{\mathcal{G}}(q,\omega) , \tag{23}$$

$$\tilde{\mathcal{G}}(q,\omega) = \frac{1}{(\omega-\eta)^2-(v_{\mathrm{F}}q)^2-|A|^2} \times \begin{pmatrix} \omega-\eta+v_{\mathrm{F}}q & A \\ A^* & \omega-\eta-v_{\mathrm{F}}q \end{pmatrix} . \tag{24}$$

Combining equations (18, 23, 24) and defining

$$\mathcal{G}(q,q',\omega) = \int\mathrm{d}x\int\mathrm{d}x'\mathrm{e}^{-\mathrm{i}(qx-q'x')}\mathcal{G}(x,x',\omega) , \tag{25}$$

we finally obtain

see equation (26) above.

The crucial observation is now that, in spite of the simple form (20) of $\Delta(x)$, it is still possible to satisfy equations (7, 8) if A and Q are interpreted as random variables. To obtain the exponential decay of the covariance we require that the probability distribution of the random momentum Q is a Lorentzian,

$$\mathcal{P}_Q = \frac{\xi}{\pi}\frac{1}{(Q\xi)^2+1} , \tag{27}$$

or equivalently for the random energy shift η defined in equation (21),

$$\mathcal{P}_\eta = \frac{\gamma}{\pi}\frac{1}{\eta^2+\gamma^2} , \tag{28}$$

with

$$\gamma = \frac{v_{\mathrm{F}}}{2\xi} . \tag{29}$$

The random variable A should be distributed such that

$$\langle A\rangle_A = 0\,, \tag{30}$$
$$\langle |A|^2\rangle_A = \Delta_{\rm s}^2\,, \tag{31}$$

where $\langle\cdots\rangle_A$ denotes averaging over the probability distribution of A. From equations (27–31) it is then easy to show that the first two moments of the distribution of $\Delta(x)$ are indeed given by equations (7, 8). Note that equations (30, 31) include the cases of pure phase and pure amplitude fluctuations. To describe pure phase fluctuations we choose $A = \Delta_{\rm s}{\rm e}^{{\rm i}\varphi}$, where the phase φ is uniformly distributed in the interval $[0, 2\pi)$. Then

$$\langle\cdots\rangle_A^{\rm ph} = \int_0^{2\pi}\frac{{\rm d}\varphi}{2\pi}\cdots \tag{32}$$

Since physical quantities should be independent of the constant phase φ and therefore should only depend on $|A|$, the process of averaging amounts to replacing $|A|$ by $\Delta_{\rm s}$. To take into account amplitude fluctuations we follow Sadovskii [10,14] and choose a Gaussian distribution for the real and imaginary parts of A,

$$\langle\cdots\rangle_A^{\rm am} = \int_{-\infty}^{\infty}\frac{{\rm dRe}A\;{\rm dIm}A}{\pi\Delta_{\rm s}^2}{\rm e}^{-|A|^2/\Delta_{\rm s}^2}\cdots \tag{33}$$

The disorder averaging of any functional $\mathcal{F}\{\Delta(x)\}$ is defined by

$$\langle\mathcal{F}\{\Delta(x)\}\rangle \equiv \left\langle\int_{-\infty}^{\infty}{\rm d}Q\,\mathcal{P}_Q\,\mathcal{F}\{A{\rm e}^{{\rm i}Qx}\}\right\rangle_A\,. \tag{34}$$

What is the physical meaning of an order parameter of the form (20)? In a superconductor such an order parameter describes a state with a uniform superflow [15]. The gauge transformation (18) corresponds to choosing a coordinate system where the superflow vanishes; η is the associated energy shift. A more detailed physical justification for such a spatially constant random energy shift η in the normal state of the cuprate superconductors has been given by Franz and Millis [16]: they pointed out that within a semiclassical approximation the effect of the quasi-static fluctuations of the phase of the order parameter field $\Delta(x)$ can be described by such an energy shift η. Franz and Millis [16] also presented a perturbative calculation of the probability distribution $\mathcal{P}_\eta$ of η, using earlier results by Emery and Kivelson [2]. Because in reference [16] a cumulant expansion of $\mathcal{P}_\eta$ was truncated at the second order, the form of $\mathcal{P}_\eta$ was found to be Gaussian by construction. However, there are certainly non-Gaussian corrections to the form of $\mathcal{P}_\eta$ given in reference [16]. Our assumption that the distribution of η is a Lorentzian of width γ is therefore not in contradiction to the work of reference [16]. Obviously, our parameter γ in equation (29) is the analog of the parameter W introduced in equation (9) of reference [16]. Note, however, that Franz and Millis [16] did not consider amplitude fluctuations of the order parameter, which are described by our second random variable A. As noted above, Gaussian amplitude fluctuations with a probability distribution given by equation (33) have been studied many years ago by Sadovskii [14]. Thus, in the present work we combine the models introduced by Sadovskii [14] and by Franz and Millis [16] such that we take both amplitude and phase fluctuations into account and still obtain an exactly solvable model.

In the following section we shall calculate a number of physical quantities for this model exactly and confirm the intuitive picture [2,16] that phase fluctuations fill in the gap at the Fermi energy and render the system metallic.

3 Calculation of physical quantities

3.1 Single-particle Green's function and spectral function

Because $\langle A\rangle = 0$, it follows from equation (26) that the off-diagonal elements of the disorder averaged Green's function vanish, and that the diagonal elements are

$$\langle\mathcal{G}_{\alpha\alpha}(q,q',\omega)\rangle = 2\pi\delta(q-q')G_\alpha(q,\omega)\,, \tag{35}$$

where

$$G_\alpha(q,\omega) = \left\langle\frac{\omega - 2\eta + \alpha v_{\rm F}q}{[\omega - 2\eta + \alpha v_{\rm F}q][\omega - \alpha v_{\rm F}q] - |A|^2}\right\rangle\,. \tag{36}$$

Here, $\alpha = +$ refers to $\mathcal{G}_{11}$, and $\alpha = -$ refers to $\mathcal{G}_{22}$. The averaging over the Lorentzian distribution (28) of the random energy shift η can be performed analytically,

$$G_\alpha(q,\omega+{\rm i}0^+) = \left\langle\frac{1}{\omega - \alpha v_{\rm F}q - \dfrac{|A|^2}{\omega + \alpha v_{\rm F}q + {\rm i}\frac{v_{\rm F}}{\xi}}}\right\rangle_A\,, \tag{37}$$

where $\langle\cdots\rangle_A$ denotes averaging over the probability distribution of A. In the case of pure phase fluctuations, as described by equation (32), this averaging is trivial, so that

$$G_\alpha^{\rm ph}(q,\omega+{\rm i}0^+) = \frac{1}{\omega - \alpha v_{\rm F}q - \Sigma_\alpha^{\rm ph}(q,\omega+{\rm i}0^+)}\,, \tag{38}$$

with the self-energy given by

$$\Sigma_\alpha^{\rm ph}(q,\omega+{\rm i}0^+) = \frac{\Delta_{\rm s}^2}{\omega + \alpha v_{\rm F}q + {\rm i}\frac{v_{\rm F}}{\xi}}\,. \tag{39}$$

Equation (39) agrees precisely with the lowest order Born approximation, which was used in the seminal work by Lee, Rice, and Anderson [5]. We have thus found a special probability distribution of $\Delta(x)$ where the lowest order Born approximation for the average single-particle Green's function is exact: the order parameter is in this case of the form $\Delta(x) = \Delta_{\rm s}{\rm e}^{{\rm i}Qx+{\rm i}\varphi}$, where Q has a Lorentzian distribution of width $1/\xi$, and the random phase φ merely assures $\langle\Delta(x)\rangle = 0$, but due to gauge invariance does not affect any physical quantities.

On the other hand, if in addition to phase fluctuations also amplitude fluctuations are important, there are

$$\Sigma_\alpha^{\mathrm{ph+am}}(q,\omega+\mathrm{i}0^+) = \cfrac{\Delta_\mathrm{s}^2}{\omega+\alpha v_\mathrm{F}q+\mathrm{i}\dfrac{v_\mathrm{F}}{\xi}-\cfrac{\Delta_\mathrm{s}^2}{\omega-\alpha v_\mathrm{F}q-\cfrac{2\Delta_\mathrm{s}^2}{\omega+\alpha v_\mathrm{F}q+\mathrm{i}\dfrac{v_\mathrm{F}}{\xi}-\cfrac{2\Delta_\mathrm{s}^2}{\omega-\alpha v_\mathrm{F}q-\cfrac{3\Delta_\mathrm{s}^2}{\omega+\alpha v_\mathrm{F}q+\mathrm{i}\frac{v_\mathrm{F}}{\xi}-\cdots}}}}} \,. \tag{41}$$

$$\Sigma_\alpha^{\mathrm{Sadovskii}}(q,\omega+\mathrm{i}0^+) = \cfrac{\Delta_\mathrm{s}^2}{\omega+\alpha v_\mathrm{F}q+\mathrm{i}\dfrac{v_\mathrm{F}}{\xi}-\cfrac{\Delta_\mathrm{s}^2}{\omega-\alpha v_\mathrm{F}q+2\mathrm{i}\dfrac{v_\mathrm{F}}{\xi}-\cfrac{2\Delta_\mathrm{s}^2}{\omega+\alpha v_\mathrm{F}q+3\mathrm{i}\dfrac{v_\mathrm{F}}{\xi}-\cfrac{2\Delta_\mathrm{s}^2}{\omega-\alpha v_\mathrm{F}q+4\mathrm{i}\dfrac{v_\mathrm{F}}{\xi}-\cfrac{3\Delta_\mathrm{s}^2}{\omega+\alpha v_\mathrm{F}q+5\mathrm{i}\frac{v_\mathrm{F}}{\xi}-\cdots}}}}} \,. \tag{42}$$

corrections to the Born approximation. For Gaussian amplitude fluctuations given by equation (33) we find after substituting $t = |A|^2/\Delta_\mathrm{s}^2$

$$G_\alpha^{\mathrm{ph+am}}(q,\omega+\mathrm{i}0^+) = \int_0^\infty \mathrm{d}t \frac{\mathrm{e}^{-t}}{\omega-\alpha v_\mathrm{F}q-\dfrac{t\Delta_\mathrm{s}^2}{\omega+\alpha v_\mathrm{F}q+\mathrm{i}\frac{v_\mathrm{F}}{\xi}}} \,. \tag{40}$$

Recently Kuchinskii and Sadovskii [17] arrived precisely at equation (40) within a diagrammatic attempt to estimate the accuracy of the method developed in reference [10] for Gaussian disorder. For a better comparison with Sadovskii's Green's function calculated in reference [10], let us represent equation (40) as a continued fraction. Expressing the integral on the right-hand side of equation (40) in terms of the incomplete Γ-function and using the known continued fraction expansion of this function [18], we obtain for the self-energy

see equation (41) above.

For the same model with Gaussian disorder the algorithm due to Sadovskii [10] produces the continued fraction expansion

see equation (42) above.

Note that only the first two lines in equations (41, 42) agree. Kuchinskii and Sadovskii argue in reference [17] that the true behavior of the Green's function for Gaussian disorder lies somewhat in between equations (41, 42). In our model, the coexistence of amplitude fluctuations with phase fluctuations (which are related to our random energy shift η) generates a completely new feature in the average spectral function. The latter is related to the average Green's function *via*

$$2\pi\delta(q-q')\langle\rho(\alpha k_\mathrm{F}+q,\omega)\rangle = -\frac{1}{\pi}\,\mathrm{Im}\,\langle\mathcal{G}_{\alpha\alpha}(q,q',\omega+\mathrm{i}0^+)\rangle \,. \tag{43}$$

Using equation (40) we find

$$\langle\rho(\alpha k_\mathrm{F}+q,\omega)\rangle^{\mathrm{ph+am}} = \frac{2\bar\gamma}{\pi\Delta_\mathrm{s}}\int_0^\infty \mathrm{d}t \frac{t\mathrm{e}^{-t}}{(t-\bar\omega^2+\bar q^2)^2+4\bar\gamma^2(\bar\omega-\alpha\bar q)^2} \,, \tag{44}$$

where $\bar q = v_\mathrm{F}q/\Delta_\mathrm{s}$, $\bar\omega = \omega/\Delta_\mathrm{s}$, and $\bar\gamma = v_\mathrm{F}/(2\Delta_\mathrm{s}\xi)$. Representative results for different values of $\bar\gamma$ are shown in Figures 1 and 2. The dashed line is the spectral function for $\bar\gamma = 0$ (*i.e.* without phase fluctuations), which is easily calculated analytically,

$$\langle\rho(\alpha k_\mathrm{F}+q,\omega)\rangle^{\mathrm{am}} = \Delta_\mathrm{s}^{-1}\Theta(\bar\omega^2-\bar q^2)|\bar\omega+\alpha\bar q|\mathrm{e}^{-(\bar\omega^2-\bar q^2)} \,. \tag{45}$$

The important point is now that for any finite $\bar\gamma$ the spectral function exhibits a logarithmic singularity at $\omega = \alpha v_\mathrm{F}q$. In the vicinity of this singularity the leading behavior of the spectral function can be calculated analytically. In the regime

$$|\omega-\alpha v_\mathrm{F}q| \ll \min\left\{\frac{\Delta_\mathrm{s}^2\xi}{v_\mathrm{F}}, \frac{\Delta_\mathrm{s}^2}{|\omega+\alpha v_\mathrm{F}q|}\right\} \tag{46}$$

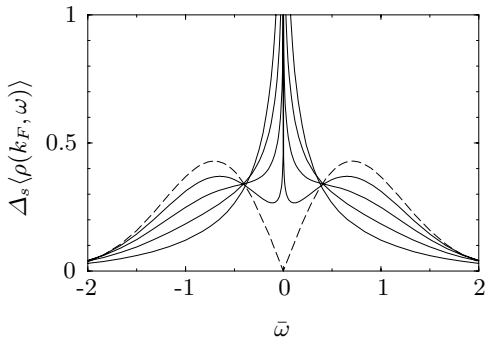

Fig. 1. Average spectral function $\langle\rho(k_{\rm F},\omega)\rangle^{\rm ph+am}$ as a function of $\bar{\omega}=\omega/\Delta_{\rm s}$ for $\bar{\gamma}=0$ (dashed line, see Eq. (45)), and $\bar{\gamma}=0.1, 0.25, 0.5, 1$ (see Eq. (44)). For any finite $\bar{\gamma}$ there is a logarithmic singularity at $\omega=0$, which acquires more weight as $\bar{\gamma}$ increases

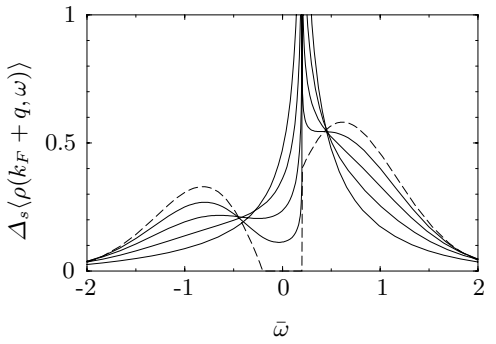

Fig. 2. Average spectral function $\langle\rho(k_{\rm F}+q,\omega)\rangle^{\rm ph+am}$ as a function of $\bar{\omega}=\omega/\Delta_{\rm s}$ for $v_{\rm F}q/\Delta_{\rm s}=0.2$. The dashed line corresponds to $\bar{\gamma}=0$ (see Eq. (45)), while the other curves correspond to $\bar{\gamma}=0.1, 0.25, 0.5, 1$ (see Eq. (44)).

the integral in equation (44) can be approximated by

$$\langle\rho(\alpha k_{\rm F}+q,\omega)\rangle^{\rm ph+am}\sim\frac{2\bar{\gamma}}{\pi\Delta_{\rm s}}\ln\left[\frac{1}{2\bar{\gamma}|\bar{\omega}-\alpha\bar{q}|}\right]=\frac{v_{\rm F}}{\pi\Delta_{\rm s}^2\xi}\ln\left[\frac{\Delta_{\rm s}^2\xi}{v_{\rm F}|\omega-\alpha v_{\rm F}q|}\right]\ . \quad (47)$$

Thus, the interplay between phase fluctuations (described by our random phase factor $\mathrm{e}^{\mathrm{i}Qx}$) and amplitude fluctuations (described by random fluctuations of $|A|$) gives rise to a logarithmic singularity at the bare energy of the electron. Note that such a singularity is weaker than the algebraic singularities that are typically found in the spectral function of a Luttinger liquid. Of course, such a weak singularity cannot be called a quasi-particle peak. It is important to point out that in the presence of amplitude fluctuations alone or phase fluctuations alone such a logarithmic singularity does not exist. Recall that for pure phase fluctuations our model has the same spectral function as predicted by the Born approximation for the self-energy [5], while for pure amplitude fluctuations our model reduces to the model discussed by Sadovskii in reference [14]. Note also that the approximate spectral function produced by Sadovskii's algorithm [19,20] for Gaussian disorder with a finite correlation length does not exhibit any logarithmic singularities. Whether an exact calculation of the spectral function for more realistic probability distributions could confirm this result or not remains an open question.

From Figure 2 it is clear that the line-shape of the spectral function in the vicinity of the singularity is rather broad and asymmetric. Such a behavior has recently been seen in the photoemission spectra of a one-dimensional band-insulator [21].

3.2 Average density of states

The average DOS is defined by

$$\langle\rho(\omega)\rangle=-\frac{1}{\pi}\,\mathrm{Im}\,\mathrm{Tr}\,\langle\mathcal{G}(x,x,\omega+\mathrm{i}0^+)\rangle\ . \quad (48)$$

Performing the q-integration in equation (37) we find

$$\mathrm{Tr}\,\langle\mathcal{G}(x,x,\omega+\mathrm{i}0^+)\rangle=-\frac{1}{v_{\rm F}}\left\langle\frac{\omega+\mathrm{i}\gamma}{\sqrt{|A|^2-(\omega+\mathrm{i}\gamma)^2}}\right\rangle_A\ , \quad (49)$$

where γ is given in equation (29) and $\sqrt{z}$ denotes the principal branch of the square root, with the cut at the negative real axis. Note that phase fluctuations simply generate an imaginary shift $\mathrm{i}\gamma$ to the frequency in equation (49). In the absence of amplitude fluctuations (see Eq. (32)) we may replace $|A|\to\Delta_{\rm s}$ in equation (49), so that we obtain for the average DOS

$$\frac{\langle\rho(\omega)\rangle^{\rm ph}}{\rho_0}=\mathrm{Im}\,\frac{z}{\sqrt{1-z^2}}\ , \quad (50)$$

where we have defined

$$z=\frac{\omega+\mathrm{i}\gamma}{\Delta_{\rm s}}=\bar{\omega}+\mathrm{i}\bar{\gamma}\ . \quad (51)$$

Equation (50) agrees exactly with the perturbative result by Lee, Rice, and Anderson [5]. For $\omega=0$ we recover equation (12). On the other hand, in the presence of additional Gaussian amplitude fluctuations, with probability distribution given by equation (33), we obtain

$$\frac{\langle\rho(\omega)\rangle^{\rm ph+am}}{\rho_0}=\mathrm{Im}\int_0^\infty\mathrm{d}t\frac{\mathrm{e}^{-t}z}{\sqrt{t-z^2}}\ . \quad (52)$$

A numerical evaluation of equation (52) is shown in Figure 3. For $\gamma=0$ the integral in equation (52) can be done analytically and reduces to the result obtained by Sadovskii [14], which does not contain phase fluctuations. In this case the DOS vanishes quadratically for small frequencies,

$$\frac{\langle\rho(\omega)\rangle^{\rm am}}{\rho_0}\sim 2\bar{\omega}^2\ \ ,\ \ |\bar{\omega}|\ll 1\ . \quad (53)$$

For any finite ξ the DOS at the Fermi energy (*i.e.* at $\omega=0$) is finite. From equation (52) we find

$$\frac{\langle\rho(0)\rangle^{\rm ph+am}}{\rho_0}=R(\bar{\gamma})\ , \quad (54)$$

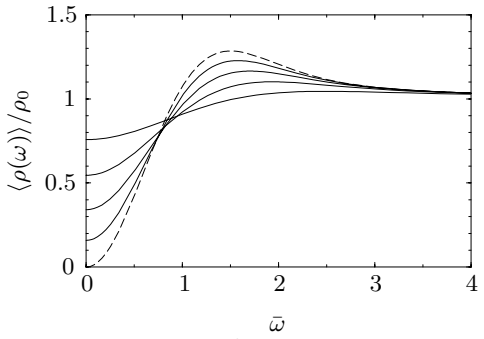

Fig. 3. Average DOS $\langle\rho(\omega)\rangle^{\rm ph+am}$ (see Eq. (52)) as a function of $\bar{\omega} = \omega/\Delta_{\rm s}$ for $\bar{\gamma} = 0$ (dashed line), and $\bar{\gamma} = 0.1, 0.25, 0.5, 1$. For smaller $\bar{\gamma}$ the pseudogap becomes deeper.

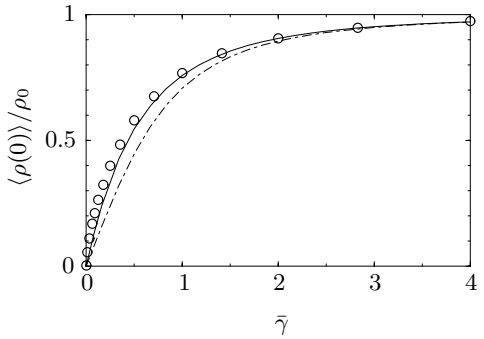

Fig. 4. Solid line: numerical evaluation of $R(\bar{\gamma}) = \langle\rho(0)\rangle^{\rm ph+am}/\rho_0$ as a function of $\bar{\gamma} = v_{\rm F}/(2\Delta_{\rm s}\xi)$ (see Eq. (55)). Dashed-dotted line: the same quantity without amplitude fluctuations (see Eq. (50)), which amounts to calculating the average DOS from the self-energy in first order Born approximation, as was done by Lee, Rice, and Anderson [5]. The circles are numerical results for Gaussian disorder, obtained *via* the exact numerical algorithm of reference [8].

with

$$R(\bar{\gamma}) = \bar{\gamma}\int_0^\infty {\rm d}t \frac{{\rm e}^{-t}}{\sqrt{t+\bar{\gamma}^2}} \ . \tag{55}$$

A numerical evaluation of $R(\bar{\gamma})$ is shown in Figure 4. For small and large $\bar{\gamma}$ we obtain to leading order

$$R(\bar{\gamma}) \sim \begin{cases} \sqrt{\pi}\bar{\gamma}, & \bar{\gamma} \ll 1 \\ 1, & \bar{\gamma} \gg 1 \end{cases} \ . \tag{56}$$

For large ξ the DOS at the Fermi energy is

$$\langle\rho(0)\rangle^{\rm ph+am} \sim \frac{\sqrt{\pi}}{2\pi\Delta_{\rm s}\xi} \ , \quad v_{\rm F}\xi \gg \Delta_{\rm s} \ , \tag{57}$$

which should be compared with the result obtained within the Born approximation, see equation (14),

$$\langle\rho(0)\rangle^{\rm pert} = \langle\rho(0)\rangle^{\rm ph} \sim \frac{1}{2\pi\Delta_{\rm s}\xi} \ . \tag{58}$$

Hence, Gaussian amplitude fluctuations increase the value of the DOS at the Fermi energy as compared with pure phase fluctuations. However, from Figure 4 it is evident that the qualitative behavior of the DOS is correctly predicted by a model with pure phase fluctuations, which exactly reproduces the perturbative result [5]. Let us emphasize that this is not the case if $\Delta(x)$ has a Gaussian distribution: the prediction of lowest order perturbation theory, $\langle\rho(0)\rangle \propto \xi^{-1}$, is in disagreement with the exact numerical result for Gaussian disorder, $\langle\rho(0)\rangle \propto \xi^{-0.64}$ (see Eq. (16)). We thus conclude that the behavior of the average DOS at the Fermi energy of the FGM in one dimension is non-universal and sensitive to the detailed form of the probability distribution of $\Delta(x)$.

3.3 Lyapunov exponent and localization length

Since the energy dispersion of the FGM is linear, the Schrödinger equation $\hat{H}_x\psi_\omega(x) = \omega\psi_\omega(x)$ is a system of linear first order differential equations. Fixing the two-component wave-function $\psi_\omega(x)$ arbitrarily at one space point x_0 therefore constitutes the wave-function at all points x. In a disordered system, the Lyapunov exponent $\kappa(\omega)$ characterizes the exponential growth of the magnitude of the wave-function at large distances $|x - x_0|$ [22],

$$|\psi_\omega(x)| \sim |\psi_\omega(x_0)| \exp[\kappa(\omega)|x - x_0|] \ . \tag{59}$$

Strictly speaking, the Lyapunov exponent is defined by the limit $|x - x_0| \to \infty$ of this equation and assumes a certain value with probability one [22]. In one dimension the inverse of the Lyapunov exponent can be identified with the *mean* localization length. According to the Thouless formula the *mean* localization length $\ell(\omega)$ can be obtained from the real part of the disorder-averaged single-particle Green's function. Originally the Thouless formula was derived for a one-band model with quadratic energy dispersion [23], but it can be shown to hold also for the FGM, where it can be written as [24,25]

$$\frac{\partial}{\partial\omega}\frac{1}{\ell(\omega)} = {\rm Re}{\rm Tr}\langle\mathcal{G}(x,x,\omega+{\rm i}0^+)\rangle \ . \tag{60}$$

Integrating the Thouless formula for equation (49), we obtain

$$\frac{v_{\rm F}}{\ell(\omega)} = {\rm Re}\left\langle\sqrt{|A|^2 - (\omega+{\rm i}\gamma)^2}\right\rangle_A - \gamma \ , \tag{61}$$

where the constant of integration is uniquely determined by the requirement $\lim_{\omega\to\infty}\ell^{-1}(\omega) = 0$. For pure phase fluctuations equation (61) reduces to

$$\frac{v_{\rm F}}{\Delta_{\rm s}\ell(\omega)^{\rm ph}} = {\rm Re}\sqrt{1 - (\bar{\omega}+{\rm i}\bar{\gamma})^2} - \bar{\gamma} \ , \tag{62}$$

while with additional Gaussian amplitude fluctuations

$$\frac{v_{\rm F}}{\Delta_{\rm s}\ell(\omega)^{\rm ph+am}} = {\rm Re}\left[\int_0^\infty {\rm d}t{\rm e}^{-t}\sqrt{t - (\bar{\omega}+{\rm i}\bar{\gamma})^2}\right] - \bar{\gamma} \ . \tag{63}$$

A plot of the inverse localization length $\ell^{-1}(\omega)^{\rm ph+am}$ is given in Figure 5. For $\gamma \to 0$ only amplitude fluctuations

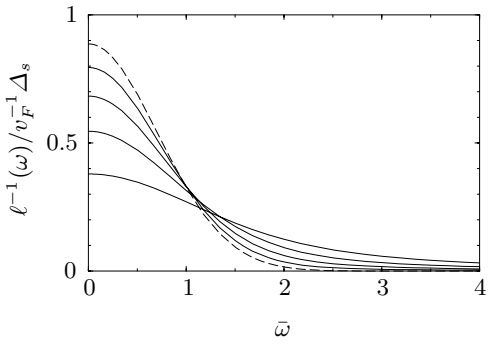

Fig. 5. Inverse localization length $\ell^{-1}(\omega)^{\rm ph+am}/v_{\rm F}^{-1}\Delta_{\rm s}$ (see Eq. (63)) as a function of $\bar{\omega} = \omega/\Delta_{\rm s}$ for $\bar{\gamma} = 0$ (dashed line), and $\bar{\gamma} = 0.1, 0.25, 0.5, 1$.

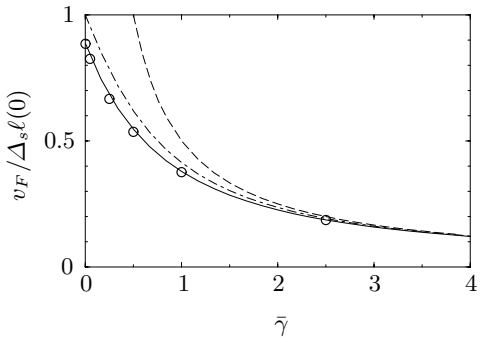

Fig. 6. Inverse localization length $P(\bar{\gamma}) = v_{\rm F}/\Delta_{\rm s}\ell(0)$ at the Fermi energy for different types of disorder. Solid line: phase and amplitude fluctuations, see equations (65, 66); dashed-dotted line: phase fluctuations, see equation (62); dashed line: extrapolation of white noise limit $P(\bar{\gamma}) = 1/(2\bar{\gamma})$, see equation (68). The circles are numerical results for Gaussian disorder, obtained *via* an exact numerical algorithm [25].

are left, and equation (63) reduces to

$$\frac{v_{\rm F}}{\Delta_{\rm s}\ell(\omega)^{\rm am}} = \frac{\sqrt{\pi}}{2}{\rm e}^{-\bar{\omega}^2} \quad , \quad \bar{\gamma} \to 0 \; . \tag{64}$$

In the presence of phase and amplitude fluctuations the general expression (63) simplifies at the Fermi energy to

$$\frac{v_{\rm F}}{\Delta_{\rm s}\ell(0)^{\rm ph+am}} \equiv P(\bar{\gamma}) \; , \tag{65}$$

where the dimensionless function $P(\bar{\gamma})$ is given by

$$P(\bar{\gamma}) = \int_0^\infty {\rm d}t {\rm e}^{-t}\left[\sqrt{t+\bar{\gamma}^2} - \bar{\gamma}\right] \; . \tag{66}$$

A comparison of equation (66) with the corresponding expression obtained from equation (62) for phase fluctuations is shown in Figure 6. For small and large $\bar{\gamma}$ the leading behavior is

$$P(\bar{\gamma}) \sim \begin{cases} \sqrt{\pi}/2, & \bar{\gamma} \ll 1 \\ 1/(2\bar{\gamma}), & \bar{\gamma} \gg 1 \end{cases} \; . \tag{67}$$

In the white noise limit $\xi \to 0$, $\Delta_{\rm s} \to \infty$ with $\Delta_{\rm s}^2\xi = {\rm const.}$ only the behavior of $P(\bar{\gamma})$ for large $\bar{\gamma}$ matters, and in this limit both equation (62) and equation (65) reduce to the known white-noise result

$$\frac{v_{\rm F}}{\ell(0)} = \frac{\Delta_{\rm s}}{2\bar{\gamma}} = \frac{\Delta_{\rm s}^2\xi}{v_{\rm F}} \; , \; \xi \to 0 \text{ with } \Delta_{\rm s}^2\xi = {\rm const.} \tag{68}$$

An extrapolation of this white-noise result towards finite correlation lengths is shown as the dashed line in Figure 6. Evidently, for large γ the behavior of the localization length becomes independent of the precise form of the probability distribution of the disorder. For $\bar{\gamma} \lesssim 1$ the localization length begins to deviate significantly from the white-noise limit and approaches a finite value of the order of $v_{\rm F}/\Delta_{\rm s}$ for $\bar{\gamma} \to 0$, the precise value of which depends on the type of the disorder. We emphasize that for a real order parameter the low-frequency behavior of the localization length is dominated by the Dyson singularity, so that in this case $1/\ell(0) = 0$ for any finite value of $\bar{\gamma}$, see references [24,25].

To compare the localization length of our exactly solvable toy model with phase and amplitude fluctuations with the case where the distribution of $\Delta(x)$ is a Gaussian, we have evaluated the Thouless formula (60) numerically for Gaussian colored noise with correlation length ξ, using an algorithm [25] similar to the one developed in reference [8]. The numerical results for $v_{\rm F}/(\Delta_{\rm s}\ell(0))$ are shown as the open circles in Figure 6. In view of the simplicity of our model the agreement with equation (65) is quite spectacular. Hence, the localization length of our model with phase and amplitude fluctuations is a very accurate approximation to the localization length of the FGM with Gaussian disorder. The dashed-dotted line in Figure 6 describes the localization length for the case where we ignore amplitude fluctuations in our model, which is equivalent to the perturbative result by Lee, Rice, and Anderson [5]. The agreement with the case of Gaussian disorder is not so good, in particular in the pseudogap regime $\bar{\gamma} \lesssim 1$.

3.4 Average conductivity

The DOS and the spectral function (see Eqs. (43, 48)) involve only the diagonal elements of the single-particle Green's function. The simplest physical quantity which involves also the off-diagonal elements of $\mathcal{G}$ is the average polarization $\langle \Pi(q, {\rm i}\omega_m)\rangle$, which is given by

$$2\pi\delta(q-q')\langle \Pi(q,{\rm i}\omega_m)\rangle = -\frac{1}{\beta}\sum_n \int \frac{{\rm d}p}{2\pi}\int\frac{{\rm d}p'}{2\pi} \times {\rm Tr}\langle \mathcal{G}(p+q,p'+q',{\rm i}\tilde{\omega}_{n+m})\mathcal{G}(p',p,{\rm i}\tilde{\omega}_n)\rangle \; . \tag{69}$$

Here, β is the inverse temperature, $\omega_m = 2\pi m/\beta$ are bosonic Matsubara frequencies and $\tilde{\omega}_n = 2\pi(n+\frac{1}{2})/\beta$ are fermionic ones. Given the average polarization, the average conductivity is easily obtained from

$$\langle \sigma(q,\omega)\rangle = -{\rm e}^2\frac{{\rm i}\omega}{q^2}\langle \Pi(q,\omega+{\rm i}0^+)\rangle \; . \tag{70}$$

In this work we shall only consider the real part of the conductivity at $q = 0$,

$$\begin{aligned}\mathrm{Re}\,\langle\sigma(\omega)\rangle &= \lim_{q\to 0}\mathrm{Re}\,\langle\sigma(q,\omega)\rangle \\ &= \mathrm{e}^2\omega\lim_{q\to 0}\frac{\langle\mathrm{Im}\,\Pi(q,\omega+\mathrm{i}0^+)\rangle}{q^2}\ . \end{aligned} \tag{71}$$

Substituting equation (26) into equation (69) and performing the Matsubara sum, we obtain for the average polarization

$$\begin{aligned}\langle\Pi(q,\mathrm{i}\omega_m)\rangle &= \Big\langle -\int\frac{\mathrm{d}p}{2\pi}\frac{E_pE_{p+q}+\xi_p\xi_{p+q}+|A|^2}{2E_pE_{p+q}} \\ &\times\left[\frac{f(E_p-\eta)-f(E_{p+q}-\eta)}{E_p-E_{p+q}-\mathrm{i}\omega_m}+\frac{f(E_p+\eta)-f(E_{p+q}+\eta)}{E_p-E_{p+q}+\mathrm{i}\omega_m}\right] \\ &+\int\frac{\mathrm{d}p}{2\pi}\frac{E_pE_{p+q}-\xi_p\xi_{p+q}-|A|^2}{2E_pE_{p+q}} \\ &\times\left[\frac{1-f(E_p-\eta)-f(E_{p+q}+\eta)}{E_p+E_{p+q}-\mathrm{i}\omega_m}\right. \\ &\left.+\frac{1-f(E_p+\eta)-f(E_{p+q}-\eta)}{E_p+E_{p+q}+\mathrm{i}\omega_m}\right]\Big\rangle\ , \end{aligned} \tag{72}$$

where we use the notation $E_p = (\xi_p^2+|A|^2)^{1/2}$, $\xi_p = v_\mathrm{F}p$ and $f(E) = 1/[\mathrm{e}^{\beta E}+1]$ is the Fermi-Dirac function. Setting $\eta = 0$ in equation (72) we recover equation (2.10) of reference [14]. Expanding equation (72) for small q and performing the average over the Lorentzian distribution of η, we obtain in the limit of zero temperature ($\beta\to\infty$),

$$\begin{aligned}\mathrm{Re}\,\langle\sigma(\omega)\rangle &= \frac{n\mathrm{e}^2}{m}\frac{\pi}{\gamma}\left\langle\sqrt{|A|^2+\gamma^2}-|A|\right\rangle_A\delta(\omega) \\ &+\frac{n\mathrm{e}^2}{m}\arctan\left(\frac{|\omega|}{\gamma}\right)\left\langle\frac{|A|^2}{\omega^2}\frac{\Theta(\omega^2-|A|^2)}{\sqrt{\omega^2-|A|^2}}\right\rangle_A\ , \end{aligned} \tag{73}$$

where $n/m \equiv v_\mathrm{F}/\pi$ and γ is defined in equation (29). For pure phase fluctuations the averaging over the distribution of A is trivial and simply leads to the replacement $|A|\to\Delta_\mathrm{s}$. Then the conductivity exhibits a Drude peak with weight given by $\bar\gamma^{-1}(\sqrt{\Delta_\mathrm{s}^2+\bar\gamma^2}-\Delta_\mathrm{s})$, which is separated from a continuum at higher frequencies by a finite gap Δ_s. Gaussian amplitude fluctuations wash out the gap but do not remove the Drude peak. Averaging over the probability distribution of the amplitude A given in equation (33) we obtain

$$\mathrm{Re}\,\langle\sigma(\omega)\rangle = \frac{n\mathrm{e}^2}{m}\left[\pi D(\bar\gamma)\delta(\omega)+\frac{1}{\Delta_\mathrm{s}}C(\bar\gamma,\bar\omega)\right]\ , \tag{74}$$

where we have used again the notation $\bar\gamma = \gamma/\Delta_\mathrm{s}$, $\bar\omega = \omega/\Delta_\mathrm{s}$, and the dimensionless functions $D(\bar\gamma)$ and $C(\bar\gamma,\bar\omega)$ are

$$D(\bar\gamma) = \frac{1}{\bar\gamma}\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t}[\sqrt{t+\bar\gamma^2}-\sqrt{t}]\ , \tag{75}$$

$$C(\bar\gamma,\bar\omega) = \arctan\left(\frac{|\bar\omega|}{\bar\gamma}\right)|\bar\omega|\int_0^1 \mathrm{d}t\mathrm{e}^{-\bar\omega^2 t}\frac{t}{\sqrt{1-t}}\ . \tag{76}$$

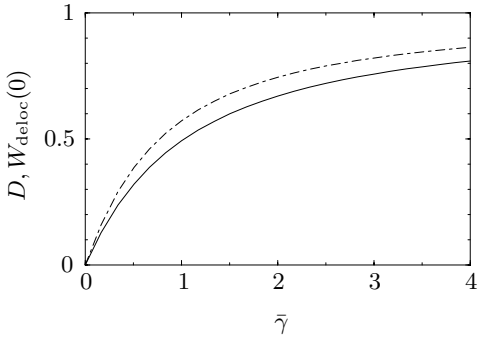

Fig. 7. Solid line: dimensionless renormalization factor D of the weight of the Drude peak as a function of $\bar\gamma = v_\mathrm{F}/(2\Delta_\mathrm{s}\xi)$, see equation (75); dashed-dotted line: probability $W_\mathrm{deloc}(0)$ for finding delocalized states at the Fermi energy, see equation (78).

A graph of $D(\bar\gamma)$ is shown in Figure 7. Physically $D(\bar\gamma)$ is the dimensionless renormalization factor for the weight of the Drude peak, with $D = 1$ corresponding to an unrenormalized Drude peak. The leading terms in the expansion of $D(\bar\gamma)$ for small and large $\bar\gamma$ are

$$D(\bar\gamma)\sim\begin{cases}\frac{\sqrt{\pi}}{2}\bar\gamma, & \bar\gamma\ll 1\\ 1, & \bar\gamma\gg 1\end{cases}\ . \tag{77}$$

At the first sight the existence of a Drude peak in our model is rather surprising because in Section 3.3 we have found that the localization length $\ell(0)$ at zero frequency is finite. In fact, we believe that for Gaussian disorder with moments given by equations (7, 8) the conductivity of the one-dimensional FGM does not exhibit a Drude peak, because the eigenstates at $\omega = 0$ should all be localized *for a given realization of the disorder* [19,22]. On the other hand, for our choice $\Delta(x) = A\mathrm{e}^{\mathrm{i}Qx}$ with spatially constant but random A and Q, the Green's function is not self-averaging, so that its spatial average is not identical with its disorder average. As a consequence, there is a finite probability of finding delocalized states at the Fermi energy: for $|\omega-\eta| > |A|$ the solutions of the Schrödinger equation are simply plane waves, whereas for $|\omega-\eta| < |A|$ there is a gap in the spectrum, and the Schrödinger equation does not have any normalizable solutions. Hence, depending on the realization of the disorder, the system is either a perfect conductor or an insulator. Because in equation (60) we have defined the inverse localization length in terms of the *disorder averaged* Green's function, the value of $\ell^{-1}(\omega)$ is determined by those realizations of the disorder where localized states at energy ω do not exist. However, the probability of finding delocalized states at the Fermi energy is finite, and can be expressed in terms of the function $P(\bar\gamma)$ defined in equation (66),

$$\begin{aligned}W_\mathrm{deloc}(0) &= \langle\Theta(\eta^2-|A|^2)\rangle \\ &= 1-\frac{2}{\sqrt{\pi}}P(\bar\gamma) \\ &\sim\begin{cases}\frac{2}{\sqrt{\pi}}\bar\gamma, & \bar\gamma\ll 1\\ 1, & \bar\gamma\gg 1\end{cases}\ . \end{aligned} \tag{78}$$

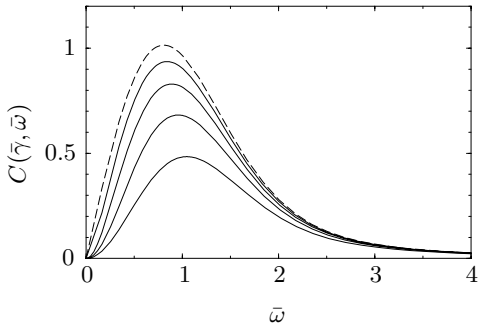

Fig. 8. Incoherent part $C(\bar{\gamma},\bar{\omega})$ of the conductivity as a function of $\bar{\omega} = \omega/\Delta_s$, see equation (76). From top to bottom: $\bar{\gamma} = 0$ (dashed line) and $\bar{\gamma} = 0.1, 0.25, 0.5, 1$.

A graph of $W_{\rm deloc}(0)$ is shown as the dashed-dotted line in Figure 7. Note that the qualitative behavior of $W_{\rm deloc}(0)$ is very similar to the weight D of the Drude peak. The conductivity of quasi-one-dimensional Peierls systems *below* the Peierls transition (for which $\langle\Delta(x)\rangle \neq 0$) has been discussed in references [26–28]. The authors pointed out that in this case a gapless collective mode associated with fluctuations of the phase of the order parameter generates a finite Drude peak.

As discussed in Section 1, our model is also relevant to describe higher-dimensional systems such as superconductors within a quasiclassical approximation. In this case it is physically reasonable to expect that phase fluctuations of the superconducting order parameter generate delocalized states at the Fermi energy [2,16]. Then we indeed expect a finite Drude peak in the conductivity, which is broadened by disorder and becomes a sharp δ-function in the superconducting state.

Let us now focus on the incoherent part of the conductivity, which is described by the dimensionless function $C(\bar{\gamma},\bar{\omega})$ in equation (74). A graph of this function is shown in Figure 8. For large correlation lengths, *i.e.* $\bar{\gamma} \ll 1$ there are three characteristic regimes where $C(\bar{\gamma},\bar{\omega})$ can be approximated by

$$C(\bar{\gamma},\bar{\omega}) \sim \begin{cases} \frac{4}{3}\bar{\gamma}^{-1}\bar{\omega}^2, & |\bar{\omega}| \ll \bar{\gamma} \\ \frac{4\pi}{6}|\bar{\omega}|, & |\bar{\gamma}| \ll |\bar{\omega}| \ll 1 \\ \frac{\pi}{2}|\bar{\omega}|^{-3}, & 1 \ll |\bar{\omega}| \end{cases} . \qquad (79)$$

For $\bar{\gamma} \ll |\bar{\omega}|$ this agrees with the result of reference [14]. Note that for a one-band model with Gaussian white noise disorder the real part of the conductivity is known to vanish for small frequencies as $\omega^2 \ln^2(1/\omega)$ [29]. Thus, apart from the logarithmic correction, the incoherent part of the conductivity of our simple model shows the generic behavior of one-dimensional disordered electrons. Note also that for small $\bar{\gamma}$ the relative weight of the Drude peak is of the order of $\bar{\gamma}$, so that the incoherent contribution dominates.

The white-noise limit is defined by letting $\Delta_s\xi \to 0$ while keeping $\Delta_s^2\xi$ finite. In this case $D(\bar{\gamma})$ approaches unity. In fact, in the white-noise limit the average conductivity is not modified by the disorder at all because the function $\Delta_s^{-1}C(\bar{\gamma},\bar{\omega})$ vanishes if we let $\Delta_s \to \infty$.

4 Conclusions

In this work we have introduced a simple exactly solvable toy model which describes the combined effects of phase and amplitude fluctuations of an off-diagonal order parameter on the physical properties of an electronic system. Although we have only discussed the one-dimensional version of this model with linearized energy dispersion, the exact solubility of our model does not depend on these features, so that our calculations can be generalized to more realistic models of electrons in dimensions $d > 1$ with non-linear energy dispersions. In this case the fluctuating gap should be chosen of the form $\Delta(\mathbf{r}) = Ae^{i\mathbf{Q}\cdot\mathbf{r}}$. To satisfy $\langle\Delta(\mathbf{r})\rangle = 0$ and $\langle\Delta(\mathbf{r})\Delta^*(\mathbf{r}')\rangle = \Delta_s^2 e^{-|\mathbf{r}-\mathbf{r}'|/\xi}$, the random variable A should be distributed such that equations (30, 31) are satisfied, while the distribution $\mathcal{P}_\mathbf{Q}$ of the d-dimensional random-vector $\mathbf{Q}$ should be

$$\mathcal{P}_\mathbf{Q} = \frac{1}{(2\pi)^d}\int d\mathbf{r} e^{-i\mathbf{Q}\cdot\mathbf{r}} e^{-|\mathbf{r}|/\xi} . \qquad (80)$$

For $d = 1$ this reduces to equation (28), but in $d > 1$ equation (80) is not a Lorentzian.

In one dimension our model describes the disordered phase of Peierls and spin-Peierls chains. We have presented explicit results for the density of states, the localization length, the single-particle spectral function, and the real part of the conductivity. Let us emphasize three points:

(a) The mean localization length of our toy model, which we have defined *via* the Thouless formula (60), is an excellent approximation to the mean localization length of the FGM with Gaussian disorder. Although the respective density of states agree quite well on a qualitative level, deviations become substantial for large correlation lengths, leading to a different scaling behavior as a function of ξ.

(b) The interplay between phase and amplitude fluctuations gives rise to a weak logarithmic singularity in the single-particle spectral function of our model. Whether this singularity is just an artifact of our toy model or not remains an open question.

(c) The conductivity of our model exhibits not only a pseudogap below the energy scale Δ_s but also a Drude peak at $\omega = 0$ with a weight that vanishes as $1/\xi$ for $\xi \to \infty$. While the qualitative picture of the continuous part should be generic for more realistic one-dimensional disordered systems (up to logarithmic corrections for small frequencies [29]), the Drude peak in our model is due to the existence of delocalized states at the Fermi energy which are created by phase fluctuations. However, in a strictly one-dimensional disordered system, the disorder should lead to the localization of all eigenstates, resulting in a vanishing zero temperature dc conductivity [29]. On the other hand, even very weak three-dimensional interactions can lead to a phase transition leading to long-range order and a finite Drude peak as found in our model. We expect that forward scattering by disorder (which we have ignored in our calculation) will broaden the Drude peak [30]. Experimentally, peak structures in the far infrared well below the pseudogap regime have been observed in the optical conductivity of several

quasi one-dimensional Peierls systems above the Peierls transition [31].

Our model also describes superconducting fluctuations in $d > 1$ within a semiclassical approximation. Recall that our equation (5) for the Green's function in $d = 1$ is formally equivalent to the Andreev equation for the semiclassical wave-function of a superconductor. The latter can be obtained from the more general Gorkov equation (1) in the limit of a slowly varying order parameter. To calculate physical observables, the solutions of the Andreev equations should be averaged over the classical trajectories of the electrons [4], which we have not done in this work. Therefore we cannot make any quantitative comparisons with experimental data for high-temperature superconductors. However, some qualitative features of our results seem to agree with experiments. In particular, in our model the pseudogap in the conductivity coexists with a small Drude peak. Such a behavior has been seen experimentally in the normal state of high-temperature superconductors [32]. In our model the Drude peak is a direct consequence of the fluctuating phase of the superconducting order parameter. Without phase fluctuations all charge carriers at the Fermi energy are localized and there is no Drude peak. In this respect our model describes a bad metal in the sense defined by Emery and Kivelson [2].

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